A MEDIAN–UNBIASED ESTIMATOR OF THE AR(1) COEFFICIENT

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ABSTRACT

A proof is given that the median of the ratios of the consecutive observations of a stationary first–order autoregressive process $X_t = \alpha X_{t-1} + Y_t$ is a median–unbiased estimator of α.

1. INTRODUCTION

The paper is concerned with the median–unbiased estimation of the stationary first order autoregressive process

$$X_t = \alpha X_{t-1} + Y_t,\ t = \dots, -1, 0, 1, \dots \tag{1}$$

with independent innovations Y_t. To the best of my knowledge there exist only two papers in the subject which contain some constructive results, namely Hurwicz (1950) and Andrews (1993). Both are concerned with the case that Y_t are i.i.d. normal $N(0,1)$ random variables.

Hurwicz (1950) observed that every ratio X_t/X_{t-1}, $t = 2, 3, \dots, n$, is a median-unbiased estimator of α. In the basic version of the process (1), where Y_t are normally distributed, the ratio X_t/X_{t-1} has a Cauchy distribution, so that neither any ratio X_t/X_{t-1} nor the mean $(n-1)^{-1}\sum_{t=2}^{n} X_t/X_{t-1}$ can be efficient. However "*one might conjecture that the median of the ratios X_t/X_{t-1}, $t = 2, 3, \dots, n$, would be a more efficient estimate of α and perhaps an unbiased one*" (Hurwicz(1950), p. 368).

Andrews (1993) constructed an exactly median–unbiased estimator of α however his proposal suffers from two disadvantages: 1) it heavy depends on the assumption of normality of innovations and 2) to apply, it needs numerical tables, separately for each number of observations used, or an appropriate computer procedure. An advantage of his approach was that the models he discussed were more general than our model (1).

The aim of this note is to prove that the Hurwicz conjecture concerning median-unbiasedness is really true. What is more, it appears that the median of the ratios is a median-unbiased estimator of α not only in the Gaussian case but whenever the medians of independent (not necessary identically distributed) innovations $Y_1, Y_2, \ldots, Y_n$ are equal to zero. It follows that the Hurwicz estimator is median-bias robust against heavy tails of innovations as well as against ε-contamination with contaminants symmetric around zero.

The problem of efficiency is more difficult first of all due to the fact that it is not as clearly stated as that of unbiasedness, and will be considered elsewhere.

2. THE HURWICZ ESTIMATOR

Our basic assumptions concerning the distributions of the innovations are that the innovations are independent, their medians are equal to zero, and they are continuous in the sense that $P\{Y_t \le 0\} = P\{Y_t \ge 0\} = 1/2$ and $P\{X_t = 0\} = 0$ for all $t = 1, 2, \ldots, n-1$; otherwise the Hurwicz estimator might not be defined.

For a given segment

$$X_1, X_2, \ldots, X_n, \quad n \text{ fixed}, \tag{2}$$

of the process (1) consider the ratios $X_2/X_1, X_3/X_2, \ldots, X_n/X_{n-1}$. To avoid too many technicalities we assume that n is even so that the median of the ratios is uniquely determined. As an estimator of α we take

$$\hat{\alpha}_{HUR} = \mathrm{Med}\left\{\frac{X_2}{X_1}, \frac{X_3}{X_2}, \ldots, \frac{X_n}{X_{n-1}}\right\} \tag{3}$$

where $\mathrm{Med}(\xi_1, \xi_2, \ldots, \xi_N)$ denotes the sample median of the observations, i.e. if $\xi_{1:N} \le \xi_{2:N} \le \ldots \le \xi_{N:N}$ and $N = 2k-1$ then $\mathrm{Med}(\xi_1, \xi_2, \ldots, \xi_N) = \xi_{k:N}$.

3. RESULTS

In the proof of the main result the following Lemma plays the central role.

Lemma. *Let $\xi_1, \xi_2, \ldots, \xi_N$, N odd, be random variables and let c be a constant such that*

(C1) $P\{\xi_j \le c\} = \dfrac{1}{2}$ *for all* $j = 1, 2, \dots, N$*;*

(C2) for every $m = 1, 2, \dots, N$*, for every choice* $i_1, i_2, \dots, i_m$ $(1 \le i_1 < i_2 < \dots < i_m \le N)$ *of integers, and for every* $x_1, \dots, x_{m-1}$

$$P\{\xi_{i_m} \le c | \xi_{i_1} = x_1, \dots, \xi_{i_{m-1}} = x_{m-1}\} = \frac{1}{2}$$

Then

$$P\{\mathrm{Med}(\xi_1, \xi_2, \dots, \xi_N) \le c\} = \frac{1}{2}.$$

Proof. First of all observe that for every $m = 1, 2, \dots, N$ and for every choice $i_1, i_2, \dots, i_m$ of different integers $1, 2, \dots, N$

$$P\{\xi_{i_1} \le c, \xi_{i_2} \le c, \dots, \xi_{i_m} \le c\} = \left(\frac{1}{2}\right)^m \tag{3}$$

That is a simple consequence of the following calculations

$$\begin{aligned}
&P\{\xi_{i_1} \le c, \xi_{i_2} \le c, \dots, \xi_{i_m} \le c\} = \\
&\quad = \int_{-\infty}^{c} \dots \int_{-\infty}^{c} P\{\xi_{i_m} \le c | \xi_{i_1} = x_1, \dots, \xi_{i_{m-1}} = x_{m-1}\} P_{\xi_{i_1} \dots \xi_{i_{m-1}}}(dx_1 \dots dx_{m-1}) \\
&\quad = \frac{1}{2} \int_{-\infty}^{c} \dots \int_{-\infty}^{c} P_{\xi_{i_1} \dots \xi_{i_{m-1}}}(dx_1 \dots dx_{m-1}) \\
&\quad = \frac{1}{2} P\{\xi_{i_1} \le c, \xi_{i_2} \le c, \dots, \xi_{i_{m-1}} \le c\}
\end{aligned}$$

where $P_{\xi_{i_1} \dots \xi_{i_{m-1}}}$ is the joint distribution of $\xi_{i_1} \dots \xi_{i_{m-1}}$.
Now we shall make use of the following formula for the distribution function of the sample median of dependent observations (David (1981), Sec. 5.6)

$$P\{\mathrm{Med}(\xi_1, \xi_2, \dots, \xi_N) \le c\} = \sum_{m=\frac{N+1}{2}}^{N} (-1)^{m - \frac{N+1}{2}} \binom{m-1}{\frac{N+1}{2} - 1} S_m$$

where S_m is the sum of $\binom{n}{m}$ probabilities $P\{\xi_{i_1} \le c, \xi_{i_2} \le c, \dots, \xi_{i_m} \le c\}$. By (3), $S_m = \binom{N}{m} \left(\frac{1}{2}\right)^m$ and hence

$$P\{\text{Med}(\xi_1,\xi_2,\dots,\xi_N) \le c\} =$$

$$= \sum_{m=\frac{N+1}{2}}^{N} (-1)^{m-\frac{N+1}{2}} \binom{m-1}{\frac{N+1}{2}-1} \binom{N}{m} \left(\frac{1}{2}\right)^m$$

$$= \frac{N!}{\left[(\frac{N+1}{2}-1)!\right]^2} \sum_{k=0}^{\frac{N+1}{2}-1} (-1)^k \binom{\frac{N+1}{2}-1}{k} \frac{1}{k+\frac{N+1}{2}} \left(\frac{1}{2}\right)^{k+\frac{N+1}{2}}$$

$$= \frac{N!}{\left[(\frac{N+1}{2}-1)!\right]^2} \sum_{k=0}^{\frac{N+1}{2}-1} (-1)^k \binom{\frac{N+1}{2}-1}{k} \int_0^{1/2} t^{k+\frac{N+1}{2}-1} dt$$

$$= \frac{N!}{\left[(\frac{N+1}{2}-1)!\right]^2} \int_0^{1/2} t^{\frac{N+1}{2}-1} \sum_{k=0}^{\frac{N+1}{2}-1} \binom{\frac{N+1}{2}-1}{k} (-t)^k dt$$

$$= \frac{N!}{\left[(\frac{N+1}{2}-1)!\right]^2} \int_0^{1/2} t^{\frac{N+1}{2}-1} (1-t)^{\frac{N+1}{2}-1} dt$$

$$= \frac{1}{2}$$

which ends the proof of the Lemma.

Theorem. *If the innovations $Y_1, Y_2, \dots, Y_n$ are independent random variables such that $P\{Y_t \le 0\} = P\{Y_t \ge 0\} = \frac{1}{2}$ for all $t = 1, 2, \dots, n$, and $P\{X_t = 0\} = 0$ for all $t = 1, 2, \dots, n-1$, then the Hurwicz estimator $\hat{\alpha}_{HUR}$ is median–unbiased:*

$$P_\alpha\{\hat{\alpha}_{HUR} \le \alpha\} = \frac{1}{2} \quad \text{for all } \alpha \in (-1, 1)$$

Proof. For the sequence of observations $X_1, X_2, \dots, X_n$, n even, denote $N = n - 1$ and apply the Lemma with

$$\xi_1 = \frac{X_2}{X_1}, \quad \xi_2 = \frac{X_3}{X_2}, \quad \dots, \quad \xi_N = \frac{X_n}{X_{n-1}}$$

For ξ_j we have

$$\xi_j = \alpha + \frac{Y_{j+1}}{X_j}$$

where Y_{j+1} and X_j are independent random variables. Now for every α

$$P_\alpha\{\xi_j \le \alpha\} = P_\alpha\{\frac{Y_{j+1}}{X_j} \le 0\}$$
$$= P_\alpha\{Y_{j+1} \le 0, X_j > 0\} + P_\alpha\{Y_{j+1} \ge 0, X_j < 0\}$$
$$= \frac{1}{2} \cdot P_\alpha\{X_j > 0\} + \frac{1}{2} \cdot P_\alpha\{X_j < 0\} = \frac{1}{2}$$

and the hypothesis (C1) of the Lemma holds.

Similarly, for every $m = 2, 3, \ldots, N$, for every choice of integers $i_1, i_2, \ldots, i_m$ $(1 \le i_1 < i_2 < \ldots < i_m \le N)$, and for every $x_1, x_2, \ldots, x_{m-1}$, taking into account that Y_{i_m+1} is independent of $X_{i_1}, \ldots, X_{i_m}$, obtains

$$P_\alpha\{\xi_{i_m} \le \alpha | \xi_{i_1} = x_1, \ldots, \xi_{i_{m-1}} = x_{m-1}\} =$$
$$= P_\alpha\{\frac{Y_{i_m+1}}{X_{i_m}} \le 0 | \xi_{i_1} = x_1, \ldots, \xi_{i_{m-1}} = x_{m-1}\}$$
$$= P_\alpha\{Y_{i_m+1} \le 0, X_{i_m} > 0 | \xi_{i_1} = x_1, \ldots, \xi_{i_{m-1}} = x_{m-1}\} +$$
$$+ P_\alpha\{Y_{i_m+1} \ge 0, X_{i_m} < 0 | \xi_{i_1} = x_1, \ldots, \xi_{i_{m-1}} = x_{m-1}\}$$
$$= \frac{1}{2} \cdot P_\alpha\{X_{i_m} > 0 | \xi_{i_1} = x_1, \ldots, \xi_{i_{m-1}} = x_{m-1}\} +$$
$$+ \frac{1}{2} \cdot P_\alpha\{X_{i_m} < 0 | \xi_{i_1} = x_1, \ldots, \xi_{i_{m-1}} = x_{m-1}\} = \frac{1}{2}$$

so that the second hypothesis (C2) of the Lemma is satisfied and the Theorem follows.

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