LOCALLY WEIBULL-SMOOTHED KAPLAN-MEIER ESTIMATOR

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ABSTRACT

Though widely used, the celebrated Kaplan-Meier estimator suffers from a disadvantage: it may happen, and in small and moderate samples it often does, that even if the difference between two consecutive times t_1 and t_2 ($t_1 < t_2$) is considerably large, for the values of the Kaplan-Meier estimators $KM(t_1)$ and $KM(t_2)$ at these times we may have $KM(t_1) = KM(t_2)$. Although that is a general problem in estimating a smooth and monotone distribution function from small or moderate samples, in the context of estimating survival probabilities the disadvantage is particularly annoying. In the paper we discuss a local smoothing of the Kaplan-Meier estimator based on an approximation by the Weibull distribution function. It appears that Mean Square Error and Mean Absolute Deviation of the smoothed estimator is significantly smaller. Also Pitman Closeness Criterion advocates for the new version of the estimator.

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INTRODUCTION

Let $F(x), x \geq 0$, be the cumulative distribution function (CDF) of time to failure X of an item and let $G(y), y \geq 0$, be the CDF of random time to censoring Y of that item. Let $T = min(X, Y)$, let $I(A)$ denote the indicator function of the set A, and let $\delta = I(X \leq Y)$. Given $t > 0$, the problem is to estimate the "survival probability" $\bar{F}(t) = 1 - F(t)$ from the "incomplete" ordered sample

$$(1) \qquad (T_1, \delta_1), (T_2, \delta_2), \ldots, (T_n, \delta_n), \quad T_1 \leq T_2 \leq \ldots, \leq T_n$$

The Kaplan-Meier (1958) estimator (KM), also called the product limit estimator, is defined as

$$(2) \qquad KM(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \dfrac{\delta_i}{n-i+1}\right)^{I(T_i \leq t)}, & \text{for } t \leq T_n \\ \begin{cases} 0, & \text{if } \delta_n = 1 \\ \text{undefined}, & \text{if } \delta_n = 0 \end{cases} & \text{for } t > T_n \end{cases}$$

In the case of ties among the T_i we adopt the usual convention that failures ($\delta_i = 1$) precede censorings ($\delta_i = 0$). By the definition, KM estimator is right-continuous.

Efron (1967) modified the estimator defining his version KMe as

$$(3) \qquad KMe(t) = \begin{cases} KM(t), & \text{if } (t > T_n \text{ and } \delta_n = 1) \text{ or } (t \leq T_n \text{ and } \delta = 0) \\ 0, & \text{otherwise} \end{cases}$$

Gill (1980) proposed another modification, we shall refer to this version by KMg, as

$$(4) \qquad KMg(t) = \begin{cases} KM(t), & \text{if } (t > T_n \text{ and } \delta_n = 1) \text{ or } (t \leq T_n) \\ KM(T_n), & \text{otherwise} \end{cases}$$

To get some intuition concerning these versions and to illustrate our approach we shall refer to the well know example from Freireich *at al.* (1963) - see also Peterson (1983) or Marubini and Valsecchi (1995). The "survival times" of 21 clinical patients were

$$6, 6, 6, 6^*, 7, 9^*, 10, 10^*, 11^*, 13, 16, 17^*, 19^*, 20^*, 22, 23, 25^*, 32^*, 32^*, 34^*, 35^* \quad (5)$$

where * denotes a censored observation. Kaplan-Meier estimator for that data is presented in Fig. 1, and Efron and Gill versions in Fig. 2.

A disadvantage of those estimators is that in small and moderate samples it may happen, and it often does, that even if the difference between two different times t_1 and t_2 ($t_1 < t_2$) is considerably large, for the values of the Kaplan-Meier estimators $KM(t_1)$ and $KM(t_2)$ at these times we may have $KM(t_1) = KM(t_2)$. For example, for the above data we have $KM(17) = KM(20) = 0.627$ and $KM(25) = KM(33) = 0.448$. It is really very difficult for a statistician to explain to a practitioner why the probability to survive at least $t = 25$ is equal to the probability of surviving at least $t = 33$! The estimator we propose, denoted by sKM, gives us $sKM(17) = 0.6402$, $sKM(20) = 0.5824$, $sKM(25) = 0.5275$, and $sKM(33) = 0.4465$ (see Fig. 3) which obviously sounds more reasonably.

Another disadvantage of the Efron and Gill estimators is that they estimate the survival probability beyond what one can reasonably conclude from the sample. It is obvious that Efron guessing will be preferable for short-tailed distributions ("a pessimistic prophet") and Gill for the fat-tailed distributions ("an optimistic prophet") but to reasonable choose between them one should restrict in a way the original nonparametric model. For that reason we confine ourselves to the original Kaplan-Meier version (2).

LOCAL WEIBULL SMOOTHING

Kaplan-Meier estimator is adequate for the nonparametric statistical model in which the only assumptions concerning possible distributions of life time are their continuity and strict monotonicity. There are some well known representatives of that family of distributions:

– exponential $E(\lambda)$ with probability density function PDF $\propto \exp\{-\lambda t\}$

– Weibull $W(\lambda, \alpha)$ with survival probability $W(t; \lambda, \alpha) = \exp\{-\lambda t^{\alpha}\}$

– gamma $\Gamma(\alpha, \lambda)$ with PDF $\propto t^{\alpha-1}\exp\{-\lambda t\})$

– generalized gamma $\Gamma_g(\lambda, \alpha, k)$ with PDF $\propto t^{\alpha k-1}\exp\{-\lambda t^{\alpha}\}$

– lognormal $logN(\mu, \sigma)$

– Gompertz $Gom(\lambda, \alpha)$ with survival probability $exp\{\lambda(1 - exp(\alpha t))\}$

– Pareto $Par(\lambda, \alpha)$ with survival probability $(1 + \lambda t)^{-\alpha}$

– log-logistic $logL(\lambda, \alpha)$ with survival probability $1/(1 + \lambda t^{\alpha})$

– exponential-power $EP(\lambda, \alpha)$ with PDF $\propto exp\{-\lambda t^{\alpha}\}$

to mention the most popular among them (e.g. Kalbfleisch and Prentice 1980, Klein et al. 1990). Here "$\propto$ means as usually "proportional to".

It is obvious that on a sufficiently short interval on the real half-line each of them my be considered as a reasonably good approximation of any CDF from the nonparametric family under consideration. We have chosen the Weibull tail $W(t; \lambda, \alpha) = exp\{-\lambda t^{\alpha}\}$ mainly because that gives us a simple algorithm of calculating the estimator: it is enough to perform logarithmic transformations of data and apply the standard estimating procedure for A and B in the simple regression model $y = Ax + B$.

On the other hand, the Weibull family $\{exp\{-\lambda t^{\alpha}\}, \lambda > 0, \alpha > 0\}$ of tails appears to be sufficiently flexible to fit all typical survival distributions. E.g. the maximal (for $t > 0$) difference between survival probabilities under gamma $\Gamma(2, 1)$ distribution and that under Weibull $W(1.522, 2.183)$ distribution is not greater than 0.010 (see Tab. 1).

Having the local approximation in mind we proceed as follows. Let $M > 1$ be a positive integer and, for a given "typical" survival distribution H, divide the positive half-line into M disjoint intervals $I(j) = [x((j-1)/M), x(j/M))$, $j = 1, 2, \ldots, M$, where $x(\beta) = H^{-1}(\beta)$ is the βth quantile (quantile of order β) of the distribution H. Let, for a fixed λ and α,

$$m_j(q) = \max_{t\in I(j)} |H(t) - W(t;\lambda,\alpha)|$$

It is obvious that, for a given H and $\varepsilon > 0$, one can find $M > 1$, and for every $j = 1, 2, \ldots, q$, one can find λ and α, such that $m_j(q) < \varepsilon$, $j = 1, 2, \ldots, q$. Tab.1 gives us the values $m_j(4)$, $j = 1, 2, 3, 4$, for a set of representatives H. It appears that if $\varepsilon = 0.01$ then $M = 4$ is enough large to ensure the local approximation within the error of ε.

If we are interested in estimating the survival probability $P\{X > t\}$ for a given t, there are two possibilities to smooth an empirical survival function (ESF) "locally". We may choose a "small" positive number $h > 0$ and approximate ESF by Weibull survival function on the interval $(t - h/2, t + h/2)$ ("a fixed window width"). Or we may fix an integer $m < n$ and approximate ESF on a random closed interval $[T_w, T_{w+m-1}]$ which contains m points ("neighbours" of t) with a suitably chosen w ("a fixed number of neighbours"). We prefer latter.

THE ESTIMATOR

Let $N-1$ be the number of distinct elements of the sample (1) in which $\delta_i = 1, i < n$, and let $i_1, i_2, \ldots, i_{N-1}$ be indexes of those elements. Let $T'_0 = 0$ and define $T'_j = T_{i_j}$, $T'_N = T_n$. Then $KM(T'_0) = 1$ and $KM(t), t < T_n$, has jumps at points $T'_j, j = 1, 2, \ldots, N-1$, and only at these points. If $\delta_n = 1$, then also $t = T_n$ is a point of a jump of KM. We shall write Kaplan-Meier estimator in the form of the sequence of pairs $(T'_j, KM'_j), j = 1, 2, \ldots, N)$, where

$$KM'_j = \begin{cases} \dfrac{KM(T'_{j-1}) + KM(T'_j)}{2}, & \text{if } j = 1, 2, \ldots, N-1 \\ \begin{cases} KM(T_n)/2 & \text{for } \delta_n = 1 \\ KM(T_n) & \text{for } \delta_n = 0 \end{cases}, & \text{if } j = N \end{cases} \tag{6}$$

Suppose we want to estimate survival probability $P\{X > t\}$ at a point t. If $t > T_n$ and $\delta_n = 0$, our estimator, like the original Kaplan-Meier estimator (2), is not defined. Otherwise we conctruct our estimator as follows.

First, choose $\varepsilon > 0$ as a satisfactory level of the error of local approximation of a survival probability by a Weibull tail and find M (see previous Section). Choose $m = [N/M]$ "neighbours" of the point t; here $[x]$ is the greatest integer smaller or equal to x. Observe that to fit a Weibull tail to m points, the numbe of points m should not be smaller than 2.

If $m = 2k$ is even, define

$$w = \begin{cases} 1, & \text{if } t < T'_k \\ j - k + 1, & \text{if } T'_{j-k+1} < \ldots < T'_j \leq t < T'_{j+1} < \ldots, < T_{j+k} \\ N - m + 1, & \text{if } T'_{N-k+1} < t \end{cases}$$

If $m = 2k + 1$ is odd, find T'_{j^*} such that $|T'_{j^*} - t| \leq |T'_j - t|$, $j = 1, 2, \ldots, N$, and define

$$w = \begin{cases} 1, & \text{if } j^* \leq k + 1 \\ j^* - k, & \text{if } k + 1 < j^* \leq N - k \\ N - m + 1, & \text{if } N - k < j^* \end{cases}$$

Take $T'_w, T'_{w+1}, \ldots, T'_{w+m-1}$ as neighbours of the point t. Then fit a Weibull tail $\exp\{-\lambda t^\alpha\}$ to them. To this end "linearize the tail" by introducing auxiliary variables

$$x_j = \log(KM'_j), \quad y_j = \log(-\log T'_j), \quad j = w, w + 1, \ldots, w + m - 1$$

and estimating regression coefficients Λ and α in

$$y = \Lambda + \alpha x$$

where $\Lambda = \log \lambda$. Finally, if $(\hat{\Lambda}, \hat{\alpha})$ are estimators of those coefficients and $\hat{\lambda} = \exp\{\hat{\Lambda}\}$, estimate survival probability $P\{X > t\}$ by

$$S(t) = \exp\{-\hat{\lambda} t^{\hat{\alpha}}\} \tag{7}$$

Like the original Kaplan-Meier estimator KM, the smoothed estimator (7) is difficult for theoretical analysis. It is however obvious that for large n and in consequence for large N (if the probability of censoring is not growing with n), and for $m = m(N)$ suitably growing with m/N bounded, the estimator $S(t)$ will behave like KM. In an asymptotic setup one can hardly expect new interesting results.

In small and moderate samples the smoothed estimator may considerably differ from the original one, in such situations however general theoretical conclusions seem to be impossible. Simulation studies (next Section) demonstrates that the proposed smoothing really improve estimation.

A SIMULATION STUDY

To compare estimators on a given set of r time-points $t_1, t_2, \ldots, t_r$ we decided to choose points of the form $t_j = x_{q_j}(H) = H^{-1}(q_j)$ with fixed $q_1, q_2, \ldots, q_r$ rather than with fixed $t_1, t_2, \ldots, t_r$ because whatever the parent distribution H the estimators at a given point $x_q(H)$ always estimate the value q. For example, if $r = 3$ and $q_1 = 0.25, q_2 = 0.5, q_3 = 0.75$, when studying the behaviour of our estimators under the exponential distribution with PDF proportional to e^{-t} we observe their values at points $0.288, 0.693, 1.386$ while under the lognormal $logN(0,1)$ distribution at points $0.509, 1.0, 1.963$. In both cases however what we estimate are the survival probabilities equal to $0.75, 0.5, 0.25$, respectively.

In all simulation studies presented below, due to our numerical experiments, we decided to choose $m = \max\{[N/M], 3\}$. All results presented in the tables and figures are based on $10,000$ simulations.

Let $MSE_{KM}(q, F, G)$ denote the mean square error of the Kaplan-Meier estimator at the point $t = x_q(F)$ if the sample comes from the distribution F and G is the censoring distribution. Similar notation $MSE_{sKM}(q, F, G)$ we apply to the smoothed version sKM of the Kaplan-Meier estimator. Analogically,

$MAD_{KM}(q,F,G)$ denotes the mean absolute deviation of the Kaplan-Meier estimator, etc.

Let $PCC(q,F,G)$ denote the Pitman Closeness Criterion (see Keating et al. 1993) for both estimators at the point $t=x_q(F)$ if F is the survival distribution and :

$$PCC(q,F,G)=P_{F,G}\{|sKM(t)-q)|\leq|KM(t)-q)|\}$$

If $PCC(q,F,G)>0.5$ then sKM prevails in the sense that the absolute error of this estimator is smaller than that of KM more often than it is larger.

Table 2 gives us ratios $\dfrac{MSE_{sKM}(q,F,G)}{MSE_{KM}(q,F,G)}$ for some survival and censoring distributions. Table 3 gives us those values for MAD, and Table 4 the values of PCC.

Fig. 4 exhibits $MSE_{sKM}(q,F,G)$, $MAD_{KM}(q,F,G))$, and PCC at the whole range of $q\in(0,1)$ for the case where the samples of size 20 or 50, respectively, come for the Weibull $F=W(1,2)$ distribution and censoring is exponential $G=E(2)$. The results presented in Fig. 4 are typical for all the results we obtained under other pairs (F,G).

ADDITIONAL COMMENTS

1. Our simulations suggest that the bias of sKM is smaller than that of KM but we were not able to find a regularity in that. Whatever however the bias and the variance, with respect to MSE and MAD smoothed sKM is better than the original KM.

2. Sufficiently far to the right, the original Kaplan-Meier estimator in all simulations gives the values zero. It follows that "practically", for large t, it's variance is equal to zero. That of course is not the case for sKM.

3. In our simulations we were also interested in MSE and other characteristics of the estimators under consideration if there is no censoring. For every $q\in(0,1)$ the Kaplan-Meier estimator is then unbiased and is variance at the point $x_q(F)$ is

equal to $q(1-q)/n$. It is interesting to observe (Fig. 5) that sometimes censoring can improve MSE. On a paradox of this kind see Csörgö et al. (1998).

4. A disadvantage of the smoothed estimator consists in that it may happen, and sometime it does (see Fig. 6), that $sKM(t_1) < sKM(t_2)$ though $t_1 < t_2$. It may happen if t_1 has a value close to the upper bound of the interval of those t, which are estimated by smoothing the points at $T'_w, T'_{w+1}, \ldots$ and t_2 is close to the lower bound of the interval of those t, which are estimated by smoothing the points at $T'_{w+1}, T'_{w+2}, \ldots$. The problem is that what we propose is not a **global** smoothing of the Kaplan-Meier estimator, but a **local** smoothing for estimating survival probability at a **given** point t, for each t separately. A kind of adjustment of estimators at two adjacent points is however needed but as yet we do not how to approach the problem.

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Tab. 1

H	$\max\lvert\bar{H}(t)-W(t;\lambda,\alpha)\rvert$				
	$(0,+\infty)$	$(0,x(1/4))$	$(x(1/4),x(1/2))$	$(x(1/2),x(3/4))$	$(x(3/4),+\infty)$
$\Gamma(2,1)$	$W(1.522,2.183)$ 0.010	$W(0.983,1.779)$ 0.006	$W(0.700,1.290)$ 0.004	$W(0.700,1.028)$ 0.004	$W(0.850,0.755)$ 0.007
$\Gamma_g(1,2,2)$	$W(0.30,3.08)$ 0.0138	$W(0.31,3.45)$ 0.0016	$W(0.309,3.112)$ 0.0022	$W(0.33,2.89)$ 0.0023	$W(0.335,2.748)$ 0.0018
$logN(0,1)$	$W(0.612,1.190)$ 0.0423	$W(0.899,1.699)$ 0.0073	$W(0.700,1.295)$ 0.0041	$W(0.700,1.028)$ 0.0036	$W(0.814,0.800)$ 0.0049
$Gom(1,1)$	$W(1.803,1.390)$ 0.0294	$W(1.300,1.092)$ 0.0025	$W(1.493,1.204)$ 0.0022	$W(1.679,1.392)$ 0.0030	$W(1.737,1.652)$ 0.0040
$Par(1,2)$	$W(1.25,0.73)$ 0.0380	$W(1.70,0.953)$ 0.0012	$W(1.55,0.905)$ 0.0027	$W(1.398,0.789)$ 0.0029	$W(1.397,0.620)$ 0.0063
$logL(1,1)$	$W(0.631,0.655)$ 0.0400	$W(0.800,0.925)$ 0.0016	$W(0.700,0.800)$ 0.0034	$W(0.700,0.635)$ 0.0050	$W(0.920,0.398)$ 0.0094
$EP(1,2)$	$W(1.907,1.301)$ 0.0216	$W(1.351,1.045)$ 0.0023	$W(1.699,1.201)$ 0.0034	$W(1.8,1.29)$ 0.0021	$W(1.862,1.448)$ 0.0020

Tab. 2

Distrib. of survival	Distrib. of censoring	q = 0.75 n			q = 0.50 n			q = 0.25 n			q = 0.10 n		
		10	20	50	10	20	50	10	20	50	10	20	50
E(1)	E(0.5)	0.81	0.84	0.84	0.79	0.84	0.83	0.66	0.72	0.71	0.62	0.61	0.58
E(1)	E(1)	0.78	0.84	0.84	0.89	0.80	0.80	0.57	0.65	0.67	0.68	0.67	0.55
E(1)	E(2)	0.74	0.79	0.84	0.58	0.70	0.74	0.53	0.61	0.58	0.78	0.74	0.57
E(2)	E(1)	0.81	0.85	0.83	0.79	0.84	0.83	0.67	0.73	0.72	0.63	0.61	0.60
E(2)	E(2)	0.78	0.85	0.84	0.70	0.80	0.81	0.57	0.65	0.67	0.67	0.65	0.56
E(2)	E(3)	0.75	0.81	0.85	0.61	0.73	0.74	0.52	0.63	0.61	0.72	0.72	0.59
W(1,2)	E(0.5)	0.78	0.84	0.82	0.77	0.84	0.83	0.67	0.77	0.74	0.59	0.61	0.61
W(1,2)	E(1)	0.74	0.81	0.82	0.68	0.78	0.80	0.55	0.65	0.68	0.60	0.60	0.55
W(1,2)	E(2)	0.75	0.73	0.83	0.60	0.60	0.79	0.49	0.53	0.69	0.63	0.64	0.57
Gom(1,1)	E(0.5)	0.82	0.83	0.82	0.81	0.84	0.84	0.69	0.77	0.77	0.55	0.56	0.55
Gom(1,1)	E(2)	0.78	0.82	0.84	0.61	0.73	0.78	0.48	0.55	0.55	0.68	0.60	0.51
Gom(1,1)	E(3)	0.79	0.74	0.82	0.54	0.59	0.70	0.47	0.51	0.53	0.83	0.79	0.61
Gom(2,1)	E(1)	0.82	0.83	0.83	0.80	0.85	0.84	0.67	0.76	0.74	0.55	0.55	0.54
Gom(2,1)	E(2.5)	0.78	0.84	0.85	0.68	0.78	0.82	0.54	0.61	0.62	0.64	0.58	0.51
Gom(2,1)	E(4)	0.78	0.81	0.84	0.60	0.72	0.74	0.48	0.57	0.57	0.73	0.65	0.55
Par(2,2)	E(1)	0.83	0.83	0.84	0.81	0.85	0.84	0.74	0.76	0.73	0.72	0.71	0.72
Par(2,2)	E(3)	0.78	0.87	0.86	0.73	0.81	0.79	0.66	0.73	0.74	0.77	0.73	0.59
Par(2,2)	E(5)	0.75	0.84	0.85	0.66	0.76	0.75	0.61	0.66	0.64	0.80	0.73	0.60
Par(2,3)	E(2)	0.82	0.85	0.83	0.81	0.84	0.84	0.75	0.74	0.72	0.74	0.68	0.67
Par(2,3)	E(4)	0.81	0.87	0.84	0.75	0.82	0.81	0.65	0.70	0.73	0.75	0.72	0.59
Par(2,3)	E(8)	0.77	0.85	0.85	0.66	0.77	0.75	0.60	0.68	0.63	0.90	0.77	0.59
LogL(5,10)	E(0.5)	0.76	0.84	0.82	0.83	0.85	0.84	0.79	0.82	0.77	0.75	0.81	0.89
LogL(5,10)	E(1)	0.66	0.80	0.82	0.77	0.81	0.84	0.76	0.81	0.76	0.72	0.73	0.83
LogL(5,10)	E(2)	0.67	0.67	0.79	0.72	0.73	0.80	0.72	0.71	0.77	0.68	0.70	0.75
LogN(0,1)	E(0.125)	0.81	0.84	0.85	0.83	0.86	0.84	0.76	0.77	0.72	0.73	0.68	0.74
LogN(0,1)	E(0.75)	0.73	0.84	0.85	0.73	0.79	0.80	0.65	0.72	0.73	0.74	0.73	0.60
LogN(0,1)	E(2)	0.73	0.71	0.82	0.65	0.66	0.74	0.71	0.64	0.66	0.86	0.79	0.75

Tab. 3

Distrib. of survival	Distrib. of censoring	$q = 0.75$ n			$q = 0.50$ n			$q = 0.25$ n			$q = 0.10$ n		
		10	20	50	10	20	50	10	20	50	10	20	50
E(1)	E(0.5)	0.89	0.92	0.92	0.89	0.92	0.91	0.81	0.85	0.83	0.73	0.74	0.75
E(1)	E(1)	0.88	0.92	0.92	0.83	0.90	0.89	0.72	0.79	0.81	0.77	0.73	0.68
E(1)	E(2)	0.86	0.89	0.91	0.76	0.84	0.86	0.66	0.72	0.72	0.81	0.77	0.67
E(2)	E(1)	0.89	0.92	0.91	0.89	0.92	0.91	0.84	0.85	0.84	0.73	0.74	0.77
E(2)	E(2)	0.88	0.92	0.92	0.84	0.89	0.90	0.73	0.79	0.82	0.76	0.72	0.68
E(2)	E(3)	0.85	0.90	0.92	0.78	0.85	0.86	0.66	0.74	0.76	0.78	0.76	0.65
W(1,2)	E(0.5)	0.88	0.91	0.91	0.88	0.92	0.91	0.82	0.87	0.86	0.73	0.76	0.79
W(1,2)	E(1)	0.85	0.90	0.91	0.82	0.89	0.89	0.73	0.81	0.82	0.74	0.73	0.72
W(1,2)	E(2)	0.87	0.85	0.91	0.75	0.78	0.88	0.66	0.68	0.83	0.77	0.76	0.74
Gom(1,1)	E(0.5)	0.89	0.91	0.91	0.90	0.92	0.92	0.83	0.88	0.88	0.71	0.73	0.74
Gom(1,1)	E(2)	0.87	0.90	0.92	0.77	0.86	0.88	0.65	0.71	0.74	0.77	0.71	0.65
Gom(1,1)	E(3)	0.91	0.85	0.90	0.72	0.76	0.84	0.61	0.65	0.67	0.83	0.83	0.69
Gom(2,1)	E(1)	0.90	0.91	0.91	0.89	0.92	0.92	0.81	0.87	0.82	0.70	0.72	0.73
Gom(2,1)	E(2.5)	0.87	0.91	0.92	0.82	0.89	0.90	0.70	0.77	0.78	0.74	0.69	0.67
Gom(2,1)	E(4)	0.87	0.90	0.92	0.77	0.85	0.86	0.64	0.71	0.74	0.79	0.73	0.65
Par(2,2)	E(1)	0.90	0.92	0.92	0.91	0.93	0.92	0.86	0.87	0.85	0.79	0.81	0.84
Par(2,2)	E(3)	0.88	0.93	0.92	0.86	0.90	0.84	0.78	0.83	0.86	0.80	0.77	0.67
Par(2,2)	E(5)	0.85	0.91	0.92	0.83	0.88	0.87	0.72	0.76	0.78	0.83	0.78	0.68
Par(2,3)	E(2)	0.89	0.93	0.91	0.90	0.92	0.91	0.85	0.86	0.84	0.78	0.78	0.82
Par(2,3)	E(4)	0.89	0.93	0.92	0.86	0.91	0.90	0.78	0.83	0.86	0.79	0.75	0.70
Par(2,3)	E(8)	0.86	0.92	0.92	0.82	0.88	0.86	0.72	0.77	0.78	0.82	0.77	0.66
LogL(5,10)	E(0.5)	0.88	0.92	0.91	0.71	0.92	0.91	0.89	0.90	0.87	0.83	0.91	0.95
LogL(5,10)	E(1)	0.81	0.90	0.91	0.88	0.90	0.92	0.87	0.89	0.86	0.80	0.84	0.92
LogL(5,10)	E(2)	0.81	0.81	0.89	0.83	0.85	0.90	0.84	0.84	0.87	0.78	0.79	0.82
LogN(0,1)	E(0.125)	0.89	0.92	0.92	0.91	0.93	0.91	0.86	0.87	0.84	0.78	0.80	0.86
LogN(0,1)	E(0.75)	0.87	0.92	0.92	0.86	0.89	0.90	0.78	0.83	0.86	0.80	0.77	0.70
LogN(0,1)	E(2)	0.85	0.84	0.91	0.79	0.82	0.86	0.80	0.74	0.75	0.91	0.86	0.80

Tab. 4

Distrib. of survival	Distrib. of censoring	$q = 0.75$ n			$q = 0.50$ n			$q = 0.25$ n			$q = 0.10$ n		
		10	20	50	10	20	50	10	20	50	10	20	50
E(1)	E(0.5)	0.63	0.59	0.59	0.59	0.59	0.60	0.65	0.62	0.62	0.85	0.75	0.66
E(1)	E(1)	0.61	0.58	0.60	0.61	0.58	0.61	0.71	0.63	0.61	0.87	0.82	0.75
E(1)	E(2)	0.60	0.59	0.60	0.64	0.57	0.60	0.86	0.73	0.64	0.91	0.91	0.93
E(2)	E(1)	0.63	0.59	0.60	0.58	0.59	0.60	0.64	0.62	0.62	0.84	0.74	0.63
E(2)	E(2)	0.61	0.59	0.60	0.60	0.59	0.60	0.71	0.63	0.61	0.87	0.84	0.76
E(2)	E(3)	0.60	0.59	0.60	0.62	0.58	0.61	0.83	0.67	0.62	0.91	0.90	0.91
W(1,2)	E(0.5)	0.61	0.60	0.60	0.59	0.60	0.60	0.65	0.61	0.62	0.84	0.73	0.64
W(1,2)	E(1)	0.62	0.60	0.60	0.61	0.59	0.61	0.73	0.65	0.63	0.87	0.81	0.70
W(1,2)	E(2)	0.62	0.61	0.61	0.69	0.62	0.60	0.86	0.75	0.64	0.89	0.86	0.71
Gom(1,1)	E(0.5)	0.53	0.59	0.64	0.58	0.59	0.59	0.64	0.61	0.59	0.82	0.72	0.67
Gom(1,1)	E(2)	0.59	0.58	0.60	0.64	0.59	0.61	0.78	0.68	0.65	0.84	0.85	0.79
Gom(1,1)	E(3)	0.58	0.60	0.60	0.70	0.60	0.59	0.92	0.80	0.69	0.86	0.83	0.87
Gom(2,1)	E(1)	0.62	0.60	0.60	0.58	0.58	0.59	0.64	0.62	0.61	0.84	0.73	0.67
Gom(2,1)	E(2.6)	0.61	0.59	0.60	0.61	0.59	0.60	0.72	0.65	0.64	0.85	0.84	0.73
Gom(2,1)	E(4)	0.59	0.59	0.60	0.63	0.60	0.61	0.81	0.69	0.64	0.85	0.87	0.84
Par(2,2)	E(1)	0.63	0.58	0.59	0.57	0.58	0.59	0.62	0.61	0.61	0.78	0.69	0.60
Par(2,2)	E(3)	0.61	0.59	0.60	0.59	0.58	0.61	0.66	0.58	0.57	0.92	0.89	0.86
Par(2,2)	E(5)	0.61	0.58	0.59	0.59	0.58	0.60	0.79	0.96	0.60	0.96	0.96	0.97
Par(2,3)	E(2)	0.64	0.57	0.60	0.58	0.59	0.60	0.63	0.61	0.61	0.81	0.71	0.61
Par(2,3)	E(4)	0.67	0.58	0.60	0.60	0.58	0.60	0.60	0.60	0.59	0.82	0.82	0.74
Par(2,3)	E(8)	0.60	0.59	0.60	0.60	0.57	0.61	0.79	0.64	0.60	0.94	0.95	0.94
LogL(5,10)	E(0.5)	0.59	0.58	0.59	0.57	0.59	0.59	0.63	0.61	0.61	0.90	0.64	0.55
LogL(5,10)	E(1)	0.65	0.59	0.59	0.60	0.58	0.60	0.66	0.61	0.62	0.86	0.74	0.59
LogL(5,10)	E(2)	0.65	0.65	0.61	0.66	0.62	0.58	0.74	0.69	0.62	0.92	0.88	0.75
LogN(0,1)	E(0.125)	0.62	0.59	0.59	0.57	0.58	0.59	0.62	0.61	0.62	0.80	0.69	0.59
LogN(0,1)	E(0.75)	0.60	0.58	0.59	0.60	0.59	0.61	0.68	0.59	0.58	0.91	0.85	0.79
LogN(0,1)	E(2)	0.63	0.61	0.59	0.71	0.57	0.56	0.95	0.86	0.70	0.99	0.99	0.98

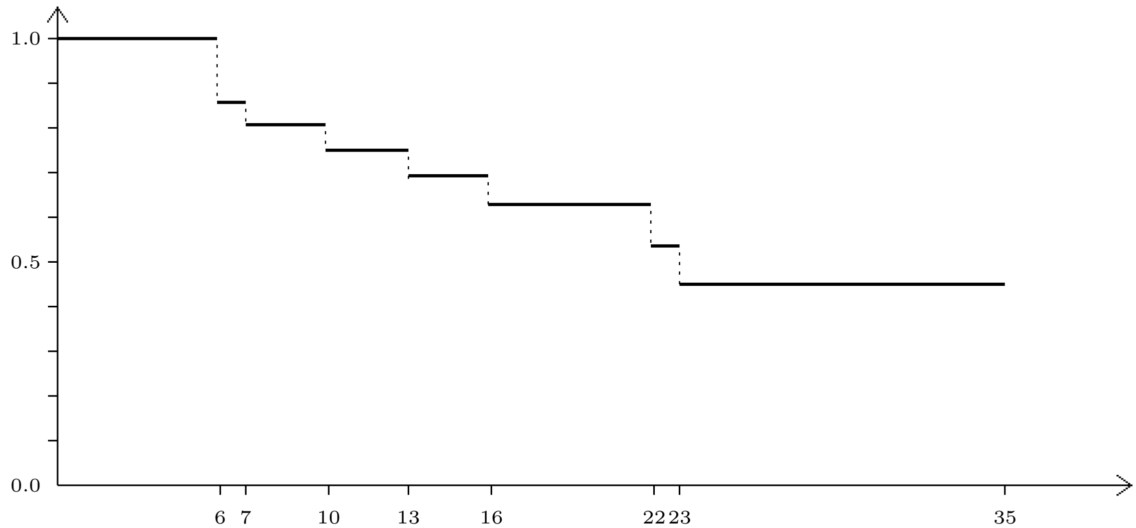

Fig.1 The Kaplan-Meier estimator KM

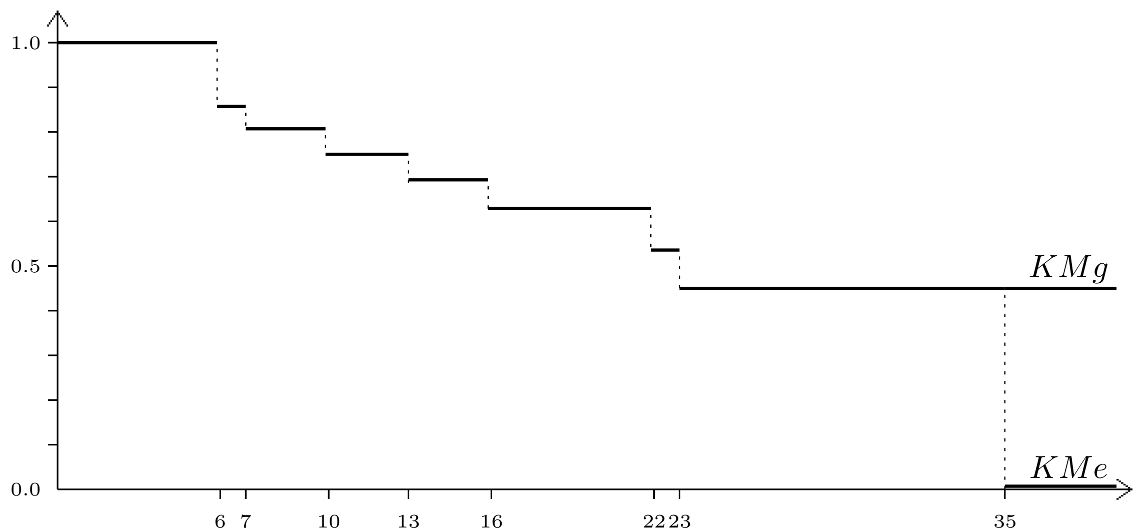

Fig.2 The Efron's and Gill's versions of the Kaplan-Meier estimator

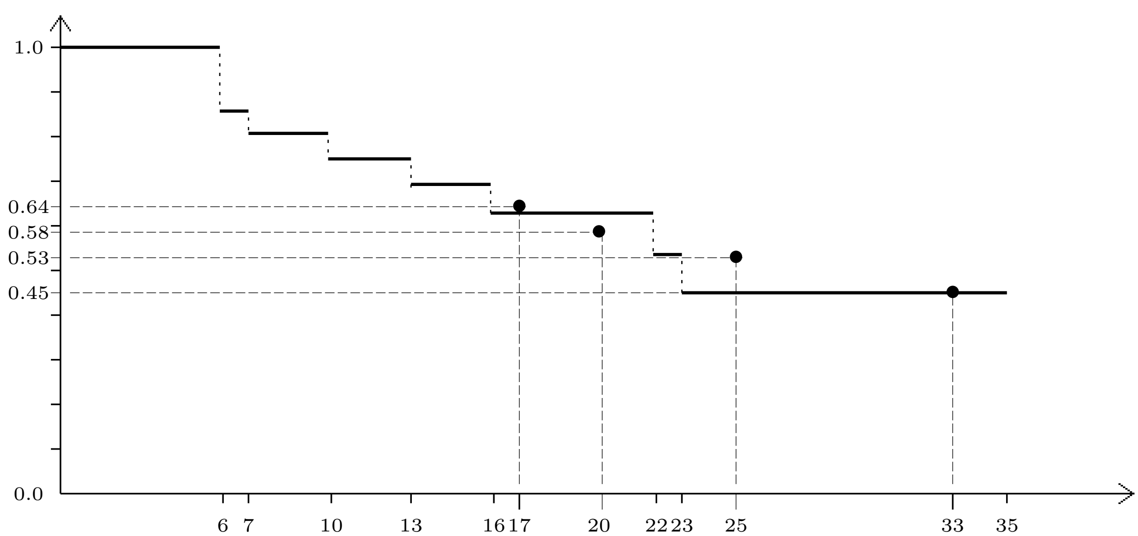

Fig. 3. Locally smoothed Kaplan-Meier estimator

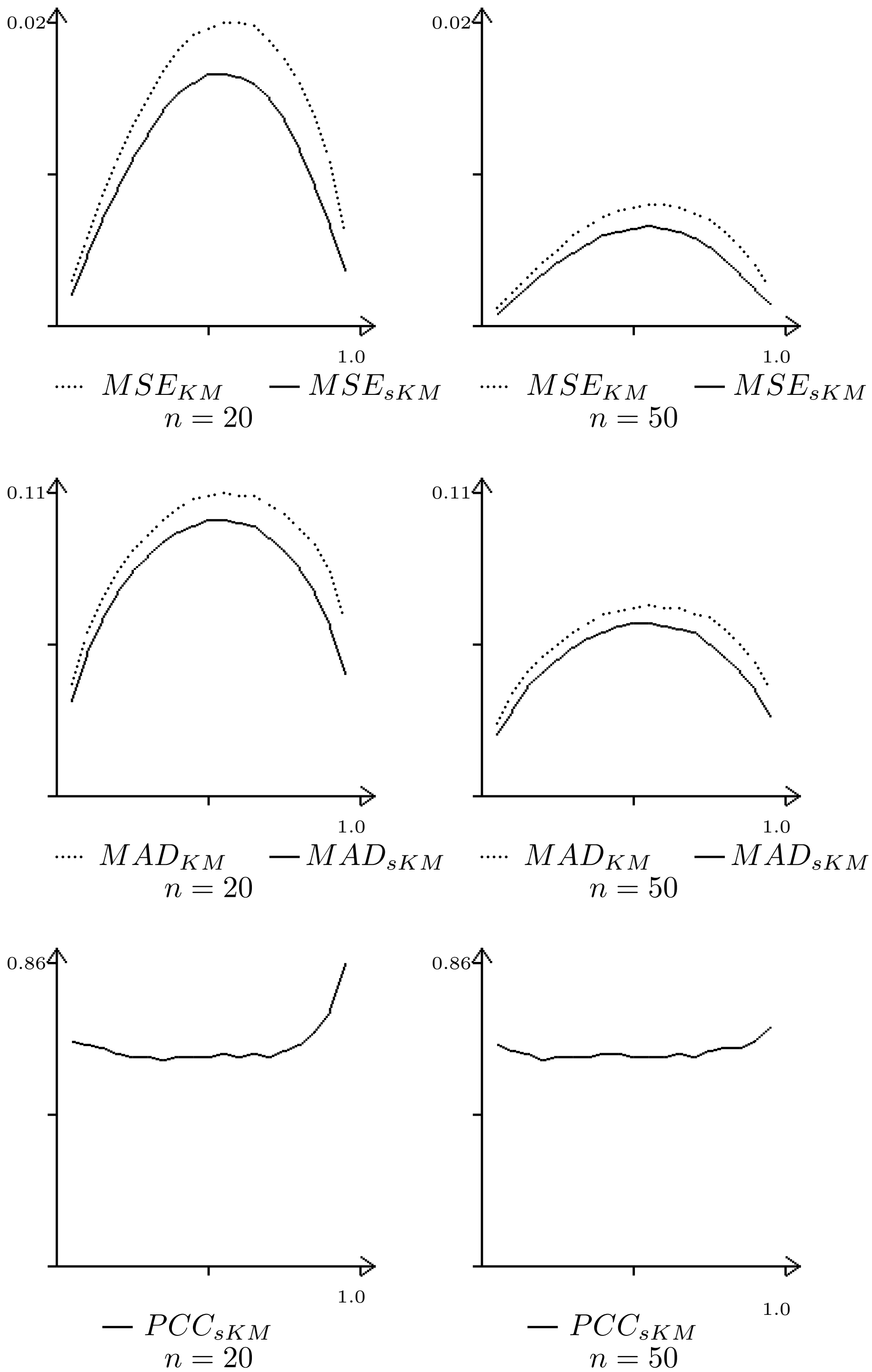

Fig. 4 Simulated MSE, MAD and PCC for the Kaplan-Meier estimator and the smoothed estimator. Sample comes from $W(1.0, 2.0)$, random censoring from $E(2)$.

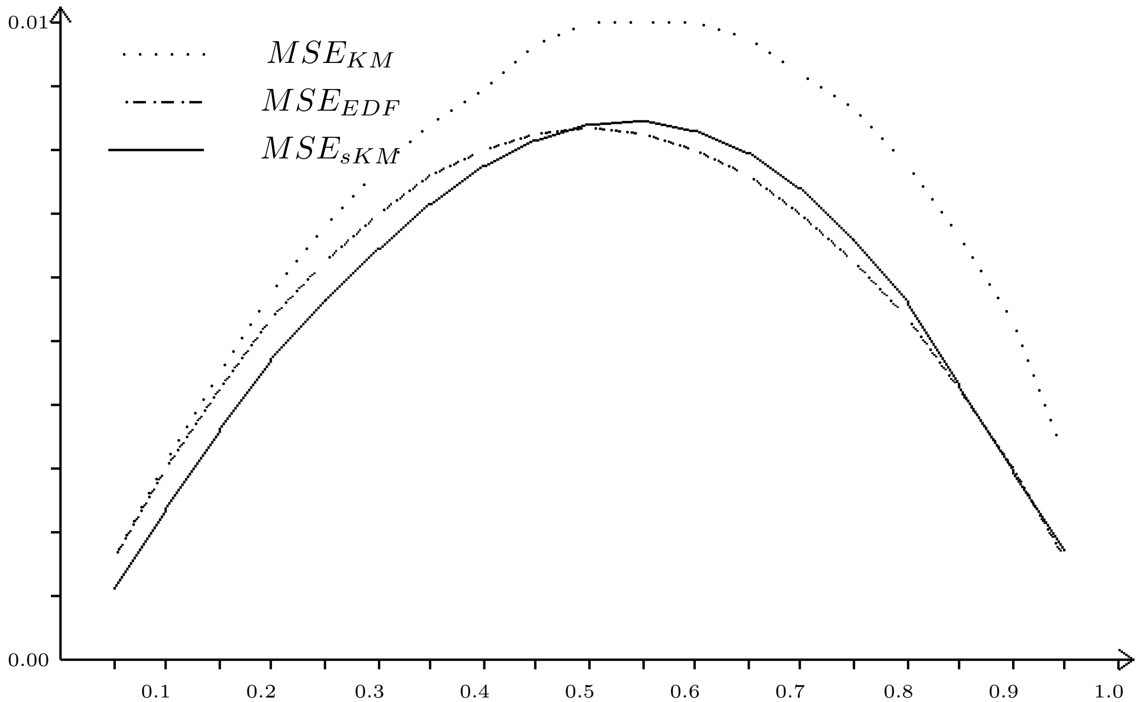

Fig.5. Simulated MSE of the Kaplan-Meier estimator and the smoothed estimator in comparison with variance of the empirical distribution function $EDF = Kaplan - Meierwithoutcensoring.$ Sample comes from $Gom(1, 1)$, random censoring from $E(2)$.

Sample size n=20.

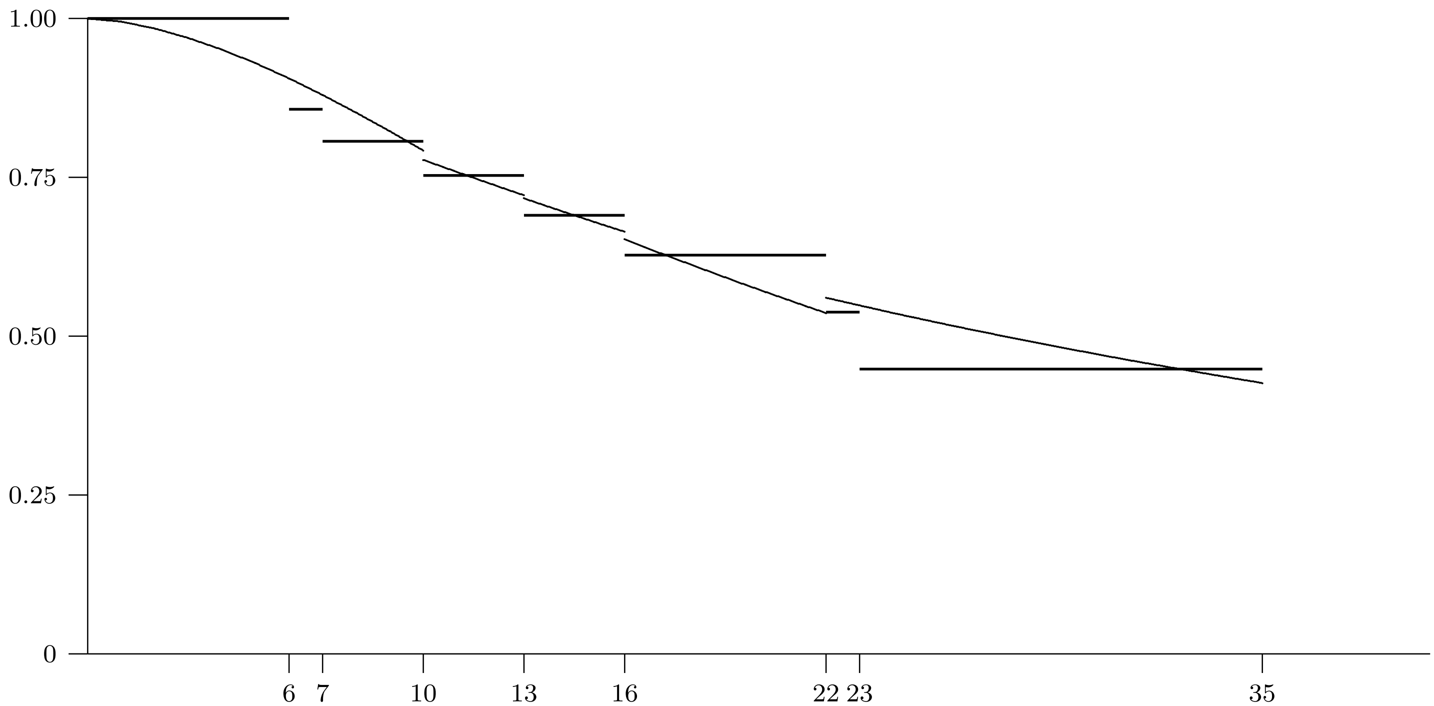

Fig.6. Kaplan-Meier and locally smoothed Kaplan-Meier estimators