ESTIMATING QUANTILES WITH LINEX LOSS FUNCTION. APPLICATIONS TO VaR ESTIMATION.

Ryszard Zieliński

Inst. Math. Polish Acad. Sc.
P.O.Box 21 Warszawa 10, Poland
e-mail: R.Zielinski@impan.gov.pl

Abstract. Sometimes, e.g. in the context of estimating VaR (*Value at Risk*), underestimating a quantile is less desirable than overestimating it which suggest to measure the error of estimation by an asymmetric loss function. As a loss function when estimating a parameter θ by an estimator T we take the well known Linex Function $exp\{\alpha(T-\theta)\} - \alpha(T-\theta) - 1$. To estimate the quantile of order $q \in (0,1)$ of a normal distribution $N(\mu, \sigma)$, we construct the optimal estimator in the class of all estimators of the form $\bar{x} + k\sigma,\ -\infty < k < \infty$, if σ is known, or of the form $\bar{x} + \lambda s$, if both parameters μ and σ are unknown; here $\bar{x}$ and s are standard estimators of μ and σ, respectively. To estimate a quantile of an unknown distribution F from the family $\mathcal{F}$ of all continuous and strictly increasing distribution functions we construct the optimal estimator in the class $\mathcal{T}$ of all estimators which are equivariant with respect to monotone transformations of data.

Mathematics Subject Classification: 62F10, 62G05

Key words and phrases: quantile estimation, Linex loss, VaR (Value at Risk), normal distribution, nonparametric model

1. The problem. In some applications underestimating a quantile is less desirable than overestimating it. That is the case, though not commonly recognized, in the problem of estimating VaR (*Value at Risk*) Khindarova at al (2000), Yi-Ping Chang at al (2003). Consequences of fixing VaR too low are essentially more serious that consequences of fixing that at a too higher level. Formally the problem of estimation of VaR may be stated as the problem of constructing the estimator which minimizes the risk of estimation under a Linex Loss function which for an estimator T and and an estimand θ takes on the form $exp\{\alpha(T-\theta)\} - \alpha(T-\theta) - 1$ (Fig. 1).

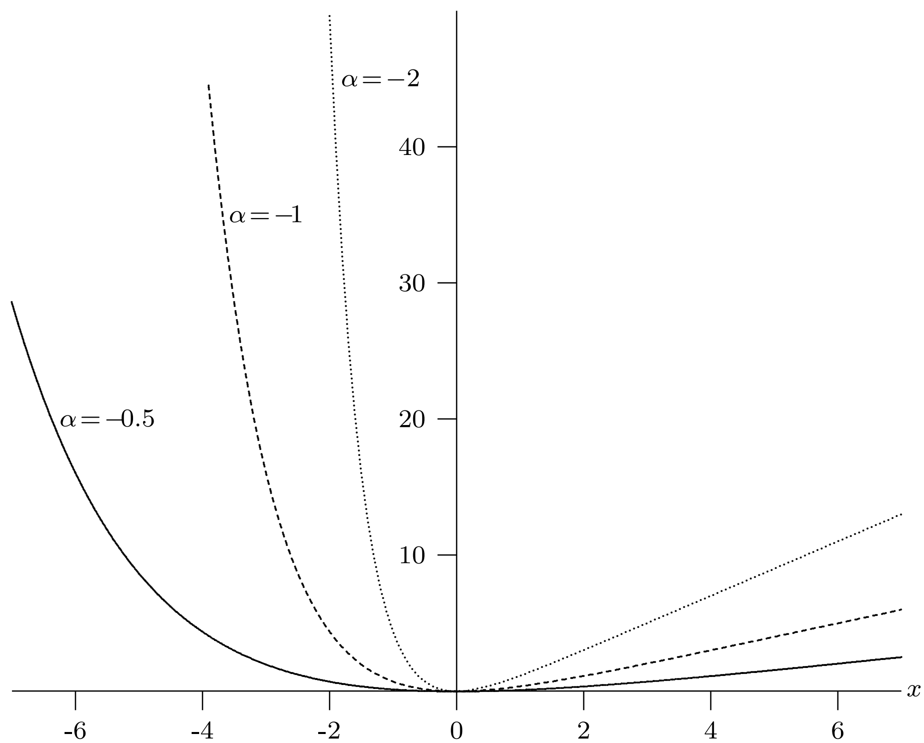

Figure 1. $Exp\{\alpha x\} - \alpha x - 1$

In what follows we construct the optimal estimator in the normal model and in a nonparametric model on the basis of a random sample $x_1, x_2, \ldots, x_n$ (i.i.d. observations) with a fixed sample size n (non-asymptotic solution).

2. Estimating quantiles of a normal distribution. Given a sample $x_1, x_2, \ldots, x_n$ from a normal distribution $N(\mu, \sigma)$, the problem is to estimate the qth quantile $x_q(\mu, \sigma) = \mu + z_q\sigma$, where $z_q = \Phi^{-1}(q)$ and Φ is the distribution function of $N(0, 1)$.

As a class of estimators we take the class of all estimators of the form $\bar{x} + k\sigma$, $-\infty < k < \infty$, if σ is known, or of the form $\bar{x} + \lambda s$, if both parameters μ and σ are unknown. Here

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j \quad \text{and} \quad s^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2$$

are standard estimators of μ and σ with probability distribution functions

$$f(\bar{x}) = \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} exp\Big\{-\frac{n}{2}\Big(\frac{\bar{x}-\mu}{\sigma}\Big)^2\Big\}$$

and

$$g(s) = \frac{2}{\sigma\Gamma(\frac{n-1}{2})}\left(\frac{n}{2}\right)^{(n-1)/2}\left(\frac{s}{\sigma}\right)^{n-2} exp\left\{-\frac{n}{2}\left(\frac{s}{\sigma}\right)^2\right\},$$

respectively.

3. Optimal estimator if σ is known. As a measure of discrepancy between the qth quantile $x_q(\mu, \sigma)$ to be estimated and the estimator $\bar{x}+k\sigma$ we take the Linex loss function in the form (Fig.1):

$$L_0(\bar{x}, k, q, n, \alpha) = exp\left\{\alpha\frac{(\bar{x}+k\sigma) - x_q(\mu,\sigma)}{\sigma}\right\} - \alpha\frac{(\bar{x}+k\sigma) - x_q(\mu,\sigma)}{\sigma} - 1.$$

Theorem 1. Assuming the loss function $L_0(\bar{x}, k, q, n, \alpha)$, the optimal estimator of the qth quantile $x_q(\mu, \sigma)$, if σ is known, is of the form

$$\bar{x} + (z_q - \alpha/2n)\sigma.$$

Proof. The risk function of the estimator $\bar{x} + k\sigma$ under the Linex loss $L_0(\bar{x}, k, q, n, \alpha)$ is given by the formula

$$R_0(k, q, n, \alpha) = \int_{-\infty}^{+\infty} L_0(\bar{x}, k, q, n, \alpha) f(\bar{x}) d\bar{x}$$

$$= exp\Big\{\alpha(k - z_q) + \frac{\alpha^2}{2n}\Big\} - \alpha(k - z_q) - 1.$$

Minimization of the risk with respect to k gives us the optimal estimator $\bar{x} + k\sigma$ with $k = k(q, n, \alpha) = z_q - \alpha/2n$. □

4. Optimal estimator if both μ and σ are unknown. As a measure of discrepancy between the qth quantile $x_q(\mu, \sigma)$ to be estimated and the estimator $\bar{x} + ks$ we take the Linex loss function in the form

$$L(\bar{x}, s, q, \lambda, n, \alpha) = exp\left\{\alpha \frac{(\bar{x} + \lambda s) - x_q(\mu, \sigma)}{\sigma}\right\} - \alpha \frac{(\bar{x} + \lambda s) - x_q(\mu, \sigma)}{\sigma} - 1.$$

Theorem 2. Assuming the loss function $L(\bar{x}, \bar{\sigma}, \lambda, q, n, \alpha)$, the optimal estimator of the qth quantile $x_q(\mu, \sigma)$, if both μ and σ are unknown, is of the form

$$\bar{x} + \lambda\bar{\sigma},$$

where $\lambda = \lambda(q, n, \alpha)$ is the unique solution of the equation

$$\int_0^{\infty} t^{n-1} exp\{\alpha\lambda t - \frac{n}{2}t^2\} dt = \frac{1}{2}\left(\frac{2}{n}\right)^{n/2} \Gamma\left(\frac{n}{2}\right) exp\left\{-\alpha\left(\frac{\alpha}{2n} - z_q\right)\right\}.$$

Comment. The left hand side of the above equation is well known as the Parabolic Cylinder Function or Weber function which is related to confluent hypergeometric functions or Whittaker functions (e.g. Abramowitz and Stegun (1972) or Gradshteyn and Ryzhik (2000)). These unable us to use standard tables or computer packages for calculating λ.

Proof. The risk function of the estimator $\bar{x} + \lambda s$ under the Linex loss $L(\bar{x}, s, q, \lambda, n, \alpha)$ is given by the formula

$$R(\lambda, q, n, \alpha) = \int_{-\infty}^{+\infty} d\bar{x} \int_0^{\infty} ds\, L(\bar{x}, s, q, \lambda, n, \alpha) f(\bar{x}) g(s)$$

Now

$$\int_{-\infty}^{\infty} exp\{\alpha\frac{\bar{x}-\mu}{\sigma}\}f(\bar{x})d\bar{x} = \frac{\sqrt{n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left\{\alpha t - \frac{n}{2}t^2\right\}dt = exp\left\{\frac{\alpha^2}{2n}\right\}$$

$$\int_{0}^{\infty} exp\left\{\alpha\frac{\lambda s - z_q\sigma}{\sigma}\right\}g(s)ds = exp\left\{-\alpha z_q\right\}\int_{0}^{\infty} exp\left\{\alpha\lambda\frac{s}{\sigma}\right\}g(s)ds$$

$$= \frac{2}{\sigma\Gamma(\frac{n-1}{2})}\left(\frac{n}{2}\right)^{\frac{n-1}{2}} exp\left\{-\alpha z_q\right\}\int_{0}^{\infty} t^{n-2}exp\left\{\alpha\lambda t - \frac{n}{2}t^2\right\}dt$$

$$\int_{-\infty}^{\infty} \alpha\frac{\bar{x}-\mu}{\sigma}f(\bar{x})d\bar{x} = 0$$

$$\int_{0}^{\infty} \alpha\left(\lambda\frac{s}{\sigma} - z_q\right)g(s)ds = \alpha\left(\lambda\sqrt{\frac{2}{n}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} - z_q\right)$$

and hence

$$R(\lambda, q,n, \alpha) =$$

$$\frac{2}{\sigma\Gamma(\frac{n-1}{2})}\left(\frac{n}{2}\right)^{\frac{n-1}{2}} exp\left\{\alpha\left(\frac{\alpha}{2n} - z_q\right)\right\}\int_{0}^{\infty} t^{n-2}exp\left\{\alpha\lambda t - \frac{n}{2}t^2\right\}dt$$

$$-\alpha\left(\sqrt{\frac{2}{n}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\lambda - z_q\right) - 1$$

The first summand of the risk $R(\lambda, q, n, \alpha)$ is a strictly decreasing function in argument λ and the second summand is a strictly increasing function so there exists exactly one λ which minimizes the risk and that is the solution of the equation $\partial R(\lambda, q, n, \alpha/\partial\lambda = 0$. □

Some numerical values of optimal k for the case of known σ and optimal λ for the case of both parameters of the parent distribution $N(0, 1)$ unknown are presented in Table 1.

Table 1. Optimal values of k (first row) and λ (second row)

q	n	α			
		-0.5	-1	-2	-5
0.5	10	0.025 0.02564	0.05 0.05133	0.1 0.10306	0.25 0.26487
	20	0.0125 0.01266	0.025 0.02532	0.05 0.05069	0.125 0.12758
	50	0.005 0.00503	0.01 0.01005	0.02 0.02010	0.05 0.05031
	100	0.0025 0.00251	0.005 0.00501	0.01 0.01003	0.025 0.02507
0.9	10	1.30655 1.36246	1.33155 1.41387	1.38155 1.52668	1.53155 1.97887
	20	1.29405 1.32117	1.30655 1.34533	1.33155 1.39582	1.40655 1.56758
	50	1.28655 1.29720	1.29155 1.30652	1.30155 1.32549	1.33155 1.38507
	100	1.28405 1.28934	1.28655 1.29395	1.29155 1.30324	1.30655 1.33174

It is obvious that $k(n,q,\alpha) \to z_q$ as $n \to \infty$. Though numerically easily confirmed, no analytic proof of the convergence $\lambda(n,q,\alpha) \to z_q$ as $n \to \infty$ is known to the author.

5. Estimating quantiles of an unknown distribution F from a large nonparametric family $\mathcal{F}$. Let $\mathcal{F}$ be the family of all continuous and strictly increasing (on their supports) distribution functions and let $x_q(F) = F^{-1}(q)$ be the (unique) q-th quantile (quantile of order q) of the distribution $F \in \mathcal{F}$. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ ($X_{1:n} \leq X_{2:n} \leq \ldots, \leq X_{n:n}$) be an ordered sample from an unknown distribution $F \in \mathcal{F}$. The sample size n is assumed to be fixed. The problem is to estimate $x_q(F)$.

As a class $\mathcal{T}$ of estimators to be considered we take the class of all estimators which are equivariant with respect to monotonic transformations of data and we measure the error of estimation of $x_q(F)$ by an estimator $T \in \mathcal{T}$ in terms of differences $F(T) - q$; rationale for the choice are to be found, for example, in Zieliński (1999, 2001, 2004). The Linex loss function takes on the form $\exp\{\alpha\,(F(T)-q)\} - \alpha\,(F(T)-q) - 1$, $\alpha < 0$ (Fig. 2).

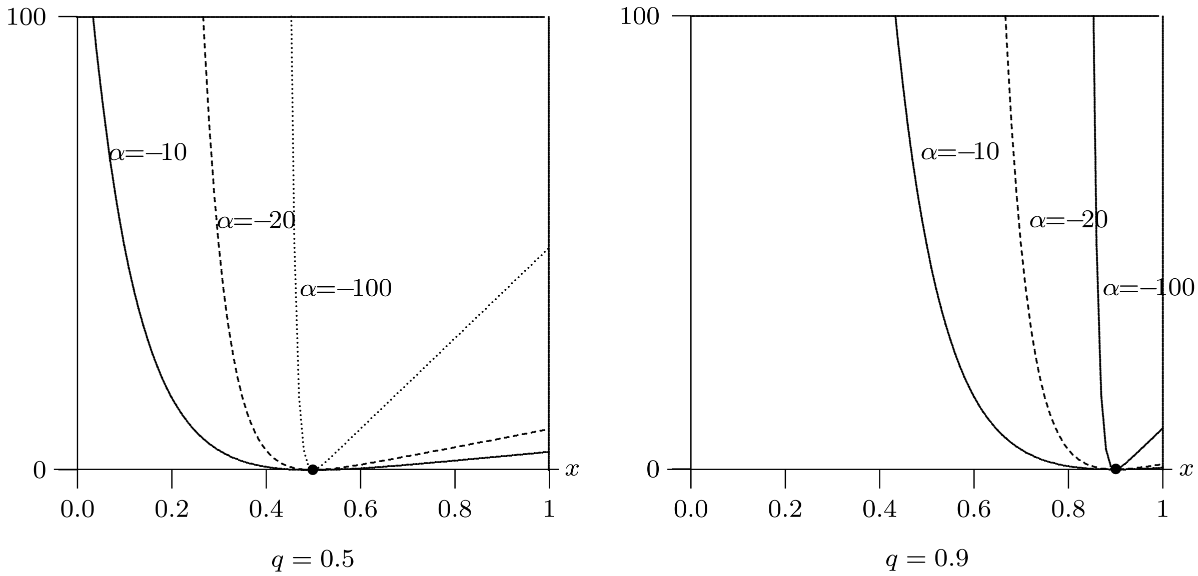

Figure 2. $Exp\{\alpha(x-q) - \alpha(x-q) - 1\}$

An estimator T belongs to the class $\mathcal{T}$ iff it is of the form $T = X_{J(\lambda):n}$, where $J = J(\lambda)$ is a random integer, independent of the sample $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, such that $P\{J = j\} = \lambda_j$, $\sum \lambda_j = 1$, $\lambda_j \geq 0$ (Uhlmann (1963) for j fixed, Zieliński (2004) for J random). Observe that if the sample comes from a distribution $F \in \mathcal{F}$ then $F(T) = F(X_{J(\lambda):n}) = U_{J(\lambda):n}$, where $U_{j:n}$ is the j-th order statistic from the uniform distribution $U(0,1)$. It follows that the risk of the estimator $T = X_{J(\lambda):n}$ under the Linex loss is given by the sum

$$\sum_{j=1}^{n} \lambda_j R(j, n; q, \alpha)$$

where

$$R(j,n;q,\alpha) =$$
$$= \frac{n!}{(j-1)!(n-j)!}\int_0^1 \Big(\exp\left\{\alpha\,(x-q)\,\right\}-\alpha(x-q)-1\Big)x^{j-1}(1-x)^{n-j}dx$$
$$= e^{-\alpha q}\,{}_1F_1(j,n+1;\alpha) - \frac{j}{n+1}\alpha + (\alpha q - 1)$$

Here

$${}_1F_1(j,n+1;\alpha) = \frac{\Gamma(n+1)}{\Gamma(j)\Gamma(n-j+1)}\int_0^1 e^{\alpha t}t^{j-1}(1-t)^{n-j}\,dt$$

is the confluent hypergeometric function (e.g. Weisstein 1999, Luke 1975). Using the recurrence relation

$${}_1F_1(j,n+1;\alpha) -_1 F_1(j-1,n+1;\alpha) = \frac{\alpha}{n+1}{}_1F_1(j,n+2;\alpha)$$

and taking into account that ${}_1F_1(j,n+1;\alpha) > 0$ we conclude that the first term in $R(j,n;q,\alpha)$ is decreasing in j, the term $j/(n+1)$ is obviously increasing and in a consequence as a result we obtain that the optimal estimator is of the form $X_{j^*:n}$ with j^* such that

$$R(j^*,n;q,\alpha) = \min_j R(j,n;q,\alpha)$$

It follows that for $j \in \{1,2,\ldots,n\}$ there exists a unique j^* such that

$$R(j^*,n;\alpha,q) < R(j,n;\alpha,q), \quad j \neq j^*$$

or

$$R(j^*,n;\alpha,q) = R(j^*+1,n;q,\alpha) < R(j,n;q,\alpha), \quad j \notin \{j^*,j^*+1\}$$

The optimal j^* can be easily found numerically. Some values of $j^* = j^*(n,\alpha,q)$ are presented in Table 2.

Table 2. Optimal $j^*(n, \alpha, q)$

$q = 0.5$ - first row, $q = 0.9$ - second row

n	α				
	-1	-10	-20	-50	-100
10	6	7	10	10	10
	10	10	10	10	10
20	11	12	13	16	20
	19	20	20	20	20
50	26	27	28	32	37
	46	47	47	49	50
100	51	52	53	57	63
	91	92	92	94	97

References

Abramowitz, M. and Stegun, I.A. (eds). (1972), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New York, Dover

Gradshteyn, I.S., Ryzhik, I.M. (2000): Tables of Integrals, Series, and Products, 6th ed., San Diego, CA, Academic Press

Khindarova, I.N. and Rachev, S.T. (2000): Value at risk: Recent advances. *In:* Anastassiou, G. (ed), Handbook of analytic-computational methods in applied mathematics, Boca Baton, Chapman & Hill/CRC

Yi-Ping Chang, Mingh-Chin Hung, and Yi-Fang Wu (2003), Nonparametric Estimation for Risk in Value-at-Risk Estimator, Communications in Statistics, Simulation and Computation, Vol.32, No.4, pp. 1041-1064

Luke, Y.L. (1975): Mathematical functions and their approsimations, Academic Press

W. Uhlmann (1963), *Ranggrössen als Schätzfunktionen*, Metrika 7 (1), 339-352.

Eric W. Weisstein (1999), *Confluent Hypergeometric Function of the First Kind*, MathWorld - A Wolfram Web Resource, http://mathworld.wolfram.com/ConfluentHypergeometricFunctionoftheFirstKind.html

R. Zieliński (1999), *Best equivariant nonparametric estimator of a quantile*, Statistics and Probability Letters 45, 79-84.

R. Zieliński (2001), *PMC-optimal nonparametric quantile estimator*, Statistics 35, 453-462.

R. Zieliński (2004), *Optimal quantile estimators; Small sample approach*, Preprint 653, IMPAN November 2004, www.impan.gov.pl/˜rziel.