

UNIFORM ASYMPTOTIC NORMALITY FOR THE BERNOULLI SCHEME

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Abstract. For every probability of success $\theta \in]0, 1[$, the sequence of Bernoulli trials is asymptotically normal, but it is *not uniformly* in $\theta \in]0, 1[$ normal. We show that the uniform asymptotic normality holds if the sequence of Bernoulli trials is randomly stopped with an appropriate stopping rule.

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1. Introduction

For the Bernoulli scheme with a probability of success θ , the central limit theorem (CLT) does *not* hold uniformly in $\theta \in]0, 1[$: for any fixed n (the number of trials), the normal approximation fails and its error is close to $1/2$ if θ is close to 0 (Zieliński 2004). CLT does not hold also for the negative Bernoulli scheme (ibid.). In our paper we show that CLT holds if n is an appropriate random variable. A sequence of stopping times and estimators are effectively constructed.

2. Main Results

Let Z_1, \dots, Z_n, \dots be a sequence of random variables defined on a statistical space with a family of distributions $\{P_\theta : \theta \in \Theta\}$.

2.1. Definition. *The sequence Z_n is uniformly asymptotically normal (UAN) if for some functions $\mu(\theta)$ and $\sigma^2(\theta)$,*

$$\forall \varepsilon \exists n_0 \forall n \geq n_0 \forall \theta \sup_{-\infty < x < \infty} \left| P_\theta \left(\frac{\sqrt{n}}{\sigma(\theta)} [Z_n - \mu(\theta)] \leq x \right) - \Phi(x) \right| < \varepsilon,$$

where Φ is the c.d.f. of the standard normal distribution $N(0, 1)$. We will then write

$$\frac{\sqrt{n}}{\sigma(\theta)} [Z_n - \mu(\theta)] \Rightarrow N(0, 1).$$

Uniform convergence in distribution is considered e.g. in Zieliński 2004, Salibian-Barrera and Zamar (2004), and Borovkov (1998). The definition above may be considered as a special case of that in Borovkov 1998.

2.2. Theorem. *Let $X = X_1, \dots, X_n, \dots$ be i.i.d. with $P_\theta(X = 1) = \theta = 1 - P_\theta(X = 0)$. The parameter space is $\Theta =]0, 1[$.*

(i) *There is no sequence of estimators $\hat{\theta}_n = \hat{\theta}_n(X_1, \dots, X_n)$ such that*

$$\frac{\sqrt{n}}{\sigma(\theta)} [\hat{\theta}_n - \theta] \Rightarrow N(0, 1).$$

(ii) *There is a sequence of stopping rules T_r ($r = 1, 2, \dots$) and a sequence of estimators $\hat{\theta}_r = \hat{\theta}_r(X_1, \dots, X_{T_r})$ such that*

$$\frac{\sqrt{r}}{\sigma(\theta)}[\hat{\theta}_r - \theta] \Rightarrow N(0, 1).$$

Proof of part (i). For every n there exists θ such that $P_\theta(X_1 = \dots = X_n = 0) > 1/2$. For such θ the probability distribution of the random variable $(\sqrt{n}/\sigma(\theta))[\hat{\theta}_n - \theta]$ has an atom which contains more than 1/2 of the total probability mass. It follows that

$$\sup_{-\infty < x < \infty} \left| P_\theta[(\sqrt{n}/\sigma(\theta))[\hat{\theta}_n - \theta] \leq x] - \Phi(x) \right| \geq 1/4.$$

□

The proof of part (ii) requires some auxiliary lemmas and will be presented in details in next sections.

3. Proofs

3.1. Lemma (A uniform version of the δ -method). *Let h be a function differentiable at μ . Assume that h and μ do not depend on θ . If*

$$V_n = \frac{\sqrt{n}}{\sigma(\theta)}[Z_n - \mu] \Rightarrow N(0, 1),$$

$h'(\mu) \neq 0$ and $\sigma(\theta) \leq b$ for some $b < \infty$ and for all $\theta \in (0, 1)$ then

$$\frac{\sqrt{n}}{\sigma(\theta)h'(\mu)}[h(Z_n) - h(\mu)] \Rightarrow N(0, 1).$$

Proof. Obviously $h(z) - h(\mu) = h'(\mu)(z - \mu) + r(z)(z - \mu)$, where $r(z) \rightarrow 0$ as $z \rightarrow \mu$, and in consequence

$$\frac{\sqrt{n}}{\sigma(\theta)h'(\mu)}[h(Z_n) - h(\mu)] = V_n + R_n$$

where

$$R_n = \frac{r(Z_n)}{h'(\mu)} \frac{\sqrt{n}}{\sigma(\theta)} [Z_n - \mu].$$

We will show that R_n tends to zero uniformly in probability P_θ , i.e. that for every $\delta > 0$,

$$(3.2) \quad \sup_{0 < \theta < 1} P_\theta(|R_n| > \delta) \rightarrow 0.$$

To this end fix $\delta > 0$ and $\varepsilon > 0$ and choose a such that $1 - \Phi(a) + \Phi(-a) < \varepsilon$. For sufficiently large n we have

$$\sup_{|z - \mu| \leq ab/\sqrt{n}} \left| \frac{r(z)}{h'(\mu)} \right| < \frac{\delta}{a}.$$

If the inequality holds then on the event $\{|V_n| \leq a\}$ we have $|Z_n - \mu| = |V_n|\sigma(\theta)/\sqrt{n} \leq ab/\sqrt{n}$ and consequently $|R_n| = |r(Z_n)/h'(\mu)| \cdot |V_n| < \delta$. For sufficiently large n we also have $\sup_\theta \sup_x |P_\theta(V_n \leq x) - \Phi(x)| < \varepsilon$ and therefore

$$\begin{aligned} \sup_\theta P_\theta(|R_n| > \delta) &\leq \sup_\theta P_\theta(|V_n| > a) \\ &\leq 1 - \Phi(a) + \Phi(-a) + 2\varepsilon < 3\varepsilon, \end{aligned}$$

which ends the proof of (3.2). We end the proof of Lemma 3.1 using the following inequalities

$$\begin{aligned} P_\theta(V_n + R_n \leq x) &\leq P_\theta(V_n \leq x + \delta) + P_\theta(|R_n| > \delta), \\ P_\theta(V_n + R_n \leq x) &\geq P_\theta(V_n \leq x - \delta) - P_\theta(|R_n| > \delta), \end{aligned}$$

and the uniform continuity of Φ . □

3.3. Berry-Esséen Theorem. By the standard Berry-Esséen Theorem for *i.i.d.* random variables Y_1, \dots, Y_n, \dots , $S_n = \sum_1^n Y_i$, and $F_n(x) = P(n^{-1/2}\sigma^{-1}[S_n - n\mu] \leq x)$ we have

$$|F_n(x) - \Phi(x)| \leq C \frac{m_3}{\sigma^3 \sqrt{n}},$$

where $m_3 = E|Y - \mu|^3$ and C is an absolute constant.

By the following sequence of inequalities $m_3^{1/3} \leq m_4^{1/4}$, $\sigma = m_2^{1/2} \leq m_4^{1/4}$, and

$$\frac{m_3}{\sigma^3} \leq \frac{m_4^{3/4}}{\sigma^3} = \frac{m_4^{3/4}}{\sigma^4} \sigma \leq \frac{m_4^{3/4}}{\sigma^4} m_4^{1/4} = \frac{m_4}{\sigma^4}$$

we obtain

3.4. Corollary

$$|F_n(x) - \Phi(x)| \leq C \frac{m_4}{\sigma^4 \sqrt{n}},$$

where $m_4 = E(Y - \mu)^4$.

Let us now consider the negative binomial scheme, that is an i.i.d. sequence of random variables geometrically distributed with the parameter θ . The central limit theorem for this scheme does not hold uniformly in $\theta \in]0, 1[$ (Zieliński 2004): the normal approximation breaks down for θ approaching 1. In the following lemma we assume θ to be bounded away from 1.

3.5. Lemma [Central Limit Theorem for the negative binomial scheme].

Let $Y = Y_1, \dots, Y_r, \dots$ be i.i.d. and let $P_\theta(Y = k) = \theta(1 - \theta)^{k-1}$ for $k = 1, 2, \dots$. Let $T_r = \sum_1^r Y_i$. Assume that $\theta \leq 1 - \kappa$: the parameter space is $\Theta =]0, 1 - \kappa]$ for some $\kappa > 0$. Then

$$\frac{\sqrt{r}}{\sqrt{1 - \theta}} \left(\frac{\theta T_r}{r} - 1 \right) \Rightarrow N(0, 1).$$

We will use following elementary facts about the geometric distribution

$$E_\theta(Y) = \frac{1}{\theta}, \quad \sigma^2(\theta) = \text{Var}_\theta(Y) = \frac{1 - \theta}{\theta^2},$$

and

$$m_4(\theta) = E_\theta(Y - \mu(\theta))^4 = \frac{(1 - \theta)(\theta^2 - 9\theta + 9)}{\theta^4}.$$

Consequently, for $\theta \leq 1 - \kappa$,

$$\frac{m_4(\theta)}{\sigma^4(\theta)} = \frac{\theta^2 - 9\theta + 9}{1 - \theta} = \frac{\theta^2}{1 - \theta} + 9 \leq \frac{1}{\kappa} + 9.$$

From Corollary 3.4 it follows that

$$\sqrt{r} \frac{\theta}{\sqrt{1-\theta}} \left(\frac{T_r}{r} - \frac{1}{\theta} \right) \Rightarrow N(0,1) \quad \text{uniformly in } \theta \in]1, 1 - \kappa].$$

□

3.6. Lemma. *Under the assumptions of the previous lemma,*

$$\frac{\sqrt{r}}{\sqrt{1-\theta}} \left(\frac{r}{\theta T_r} - 1 \right) \Rightarrow N(0,1).$$

Proof. It is enough to combine Lemma 3.6 with Lemma 3.1 (δ -method) applied to the function $h(x) = 1/x$ at $\mu = 1$. □

3.7. Lemma. *Let X_1, \dots, X_n, \dots be the Bernoulli scheme with a probability of success θ . Define the sequence of stopping rules $T'_r = \min\{n : S_n \geq r\}$, where $S_n = \sum_1^n X_i$. The sequence $\hat{\theta}'_r = r/T'_r$ is UAN in $\theta \leq 1 - \kappa$, i.e. for the parameter space $\Theta =]0, 1 - \kappa]$.*

Proof. This is a simple reformulation of Lemma 3.6. Indeed, it is easy to see that T'_r is a sum of i.i.d. geometrically distributed random variables.

Proof of Theorem 2.2(ii). The sequence of stopping times $T_r, r = 1, 2, \dots$, will be constructed as follows. Define $T'_r = \min\{n : S_n \geq r\}$, $T''_r = \min\{n : n - S_n \geq r\}$,

$$\tilde{T}_r = \min\{n : S_n \geq r, n - S_n \geq r\} = \max(T'_r, T''_r),$$

and

$$T_r = \tilde{T}_r + r.$$

The sequence of estimators $\hat{\theta}_r$ will be constructed as follows. Define two auxiliary estimators $\hat{\theta}'_r = r/T'_r$ and $\hat{\theta}''_r = 1 - r/T''_r$, a random event

$$A_r = \left\{ \frac{1}{r} \sum_{i=1}^r X_{\tilde{T}_r+i} < \frac{1}{2} \right\},$$

and finally

$$\hat{\theta}_r = \begin{cases} \hat{\theta}'_r & \text{on } A_r \\ \hat{\theta}''_r & \text{on } A_r^c. \end{cases}$$

We claim that $\hat{\theta}_r$ is UAN on $]0, 1[$ with the asymptotic variance $\sigma^2(\theta)$ given by the formula:

$$\sigma^2(\theta) = \begin{cases} (1-\theta)\theta^2 & \text{for } \theta < 1/2, \\ (1-\theta)^2\theta & \text{for } \theta \geq 1/2. \end{cases}$$

To prove that fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\sup_{1/2-\delta < \theta < 1/2+\delta} \sup_x \left| \Phi\left(\frac{x}{\theta\sqrt{1-\theta}}\right) - \Phi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right) \right| < \varepsilon.$$

Obviously $\delta < 1/2$.

Choose r_1 such that for $r \geq r_1$ the inequality $P_\theta(A_r^c) < \varepsilon$ holds for all $\theta < 1/2 - \delta$ and $P_\theta(A_r) < \varepsilon$ holds for all $\theta > 1/2 + \delta$.

From Lemma 3.7 we conclude that

$$\frac{\sqrt{r}}{\theta\sqrt{1-\theta}} (\hat{\theta}'_r - \theta) \Rightarrow N(0, 1) \quad \text{on }]0, 1/2 + \delta]$$

and

$$\frac{\sqrt{r}}{\sqrt{\theta}(1-\theta)} (\hat{\theta}''_r - \theta) \Rightarrow N(0, 1) \quad \text{on } [1/2 - \delta, 1[.$$

Choose r_2 such that for $r \geq r_2$ and for all $\theta \leq 1/2 + \delta$,

$$\begin{aligned} & \sup_x \left| P_\theta \left(\sqrt{r} \frac{\hat{\theta}'_r - \theta}{\theta\sqrt{1-\theta}} \leq x \right) - \Phi(x) \right| \\ &= \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}'_r - \theta) \leq x \right) - \Phi\left(\frac{x}{\theta\sqrt{1-\theta}}\right) \right| < \varepsilon. \end{aligned}$$

Then for $r \geq r_2$ and for all $\theta \geq 1/2 - \delta$ we also have

$$\begin{aligned} & \sup_x \left| P_\theta \left(\sqrt{r} \frac{\hat{\theta}''_r - \theta}{\sqrt{\theta}(1-\theta)} \leq x \right) - \Phi(x) \right| \\ &= \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}''_r - \theta) \leq x \right) - \Phi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right) \right| < \varepsilon. \end{aligned}$$

Define $r_0 = \max(r_1, r_2)$.

For the estimator $\hat{\theta}_r$ we obtain

$$\begin{aligned} & \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\ & \leq \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}_r - \theta) \leq x, A_r \right) - P_\theta(A_r) \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\ & \quad + \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}_r - \theta) \leq x, A_r^c \right) - P_\theta(A_r^c) \Phi \left(\frac{x}{\sigma(\theta)} \right) \right|. \end{aligned}$$

Due to the facts that $\hat{\theta}_r = \hat{\theta}'_r$ on A_r and $\hat{\theta}'_r$ and A_r are independent, and similarly $\hat{\theta}_r = \hat{\theta}''_r$ on A_r^c and $\hat{\theta}''_r$ and A_r^c are independent, the Right Hand Side of the latter formula is equal to

$$\begin{aligned} & P_\theta(A_r) \cdot \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}'_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\ & \quad + P_\theta(A_r^c) \cdot \sup_x \left| P_\theta \left(\sqrt{r}(\hat{\theta}''_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right|. \end{aligned}$$

For $\theta < 1/2 - \delta < 1/2$ we have $P_\theta(A_r^c) < \varepsilon$, $\sigma^2(\theta) = (1 - \theta)\theta^2$, and

$$\left| P_\theta \left(\sqrt{r}(\hat{\theta}'_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\theta\sqrt{1-\theta}} \right) \right| < \varepsilon.$$

For $\theta > 1/2 + \delta > 1/2$ we have $P_\theta(A_r) < \varepsilon$, $\sigma^2(\theta) = (1 - \theta)^2\theta$, and

$$\left| P_\theta \left(\sqrt{r}(\hat{\theta}''_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\sqrt{\theta}(1-\theta)} \right) \right| < \varepsilon.$$

For $1/2 - \delta < \theta < 1/2 + \delta$

$$\begin{aligned} & \left| P_\theta \left(\sqrt{r}(\hat{\theta}'_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\ & < \left| P_\theta \left(\sqrt{r}(\hat{\theta}'_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\theta\sqrt{1-\theta}} \right) \right| + \left| \Phi \left(\frac{x}{\theta\sqrt{1-\theta}} \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\ & < 2\varepsilon \end{aligned}$$

and similarly

$$\begin{aligned}
& \left| P_{\theta} \left(\sqrt{r}(\hat{\theta}_r'' - \theta) \leq x \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\
& < \left| P_{\theta} \left(\sqrt{r}(\hat{\theta}_r'' - \theta) \leq x \right) - \Phi \left(\frac{x}{\sqrt{\theta}(1-\theta)} \right) \right| + \left| \Phi \left(\frac{x}{\sqrt{\theta}(1-\theta)} \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| \\
& < 2\varepsilon.
\end{aligned}$$

Eventually we obtain

$$\sup_x \left| P_{\theta} \left(\sqrt{r}(\hat{\theta}_r - \theta) \leq x \right) - \Phi \left(\frac{x}{\sigma(\theta)} \right) \right| < 4\varepsilon$$

which ends the proof. □

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