Elementy statystyki bayesowskiej

Model statystyczny

$$(\mathcal{X}, \{P_\theta, \theta \in \Theta\})$$

Rozkład *a priori*:
rozkład prawdopodobieństwa Π na zbiorze Θ

Rozkład P_θ oznaczać będziemy $P(\cdot|\theta)$, a jego gęstość przez $p(\cdot|\theta)$

Niech $\pi(\cdot)$ oznacza gęstość rozkładu *a priori*

Rozkład *a posteriori*:

$$\pi(\theta|x) = \pi(\theta)\frac{p(x|\theta)}{p(x)}, \qquad p(x) = \int_\Theta \pi(\theta)p(x|\theta)d\theta$$

Estymacja bayesowska

Funkcja straty: $L(\hat{\theta}, \theta)$

Ryzyko bayesowskie:

$$\int_{\Theta} \left[\int_{\mathcal{X}} L(\hat{\theta}(x), \theta) p(x|\theta) dx \right] \pi(\theta) d\theta$$
$$= \int_{\mathcal{X}} \left[\int_{\Theta} L(\hat{\theta}(x), \theta) \pi(\theta|x) d\theta \right] p(x) dx$$

Estymator bayesowski minimalizuje

$$\int_{\Theta} L(\hat{\theta}(x), \theta) \pi(\theta|x) d\theta$$

Jeżeli $L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, to estymatorem bayesowskim jest wartość oczekiwana rozkładu *a posteriori*

Bayesowskie testowanie hipotez

Zbiór decyzji

$$\mathcal{D} = \{d_1 = \{\theta = \theta_1\}, d_2 = \{\theta = \theta_2\}\}$$

Funkcja straty $L(d, \theta)$

	$\theta = \theta_1$	$\theta = \theta_2$
d_1	L_{11}	L_{12}
d_2	L_{21}	L_{22}

Rozkład *a priori*

$$P\{\theta = \theta_1\} = \pi_1 \quad P\{\theta = \theta_2\} = \pi_2 \quad \pi_1 + \pi_2 = 1$$

Oczekiwana strata *a posteriori* reguły δ

$$\begin{aligned} \delta(x) = d_1 : &\quad \pi_1 p(x|\theta_1) L_{11} + \pi_2 p(x|\theta_2) L_{12} \\ \delta(x) = d_2 : &\quad \pi_1 p(x|\theta_1) L_{21} + \pi_2 p(x|\theta_2) L_{22} \end{aligned}$$

Reguła bayesowska: obszar przyjęć decyzji d_2

$$\left\{ x \in \mathcal{X} : \frac{p(x|\theta_2)}{p(x|\theta_1)} > \frac{\pi_1 [L_{21} - L_{11}]}{\pi_2 [L_{12} - L_{22}]} \right\}$$

Przykład (Model dwumianowy).

Model pojedynczej obserwacji X:

$$(\{0,1\}, \{D(\theta),\ \theta \in [0,1]\})$$

Rozkład *a priori* $U(0,1)$: $\pi(\theta) = 1$ $(\theta \in [0,1])$

Rozkład *a posteriori* $(n = 1)$

$$P(1) = \int_0^1 P(1|\theta)\pi(\theta)d\theta = \int_0^1 \theta d\theta = \frac{1}{2}$$

$$P(0) = \int_0^1 P(0|\theta)\pi(\theta)d\theta = \int_0^1 (1-\theta)d\theta = \frac{1}{2}$$

$$\pi(\theta|1) = \frac{P(1|\theta)\pi(\theta)}{P(1)} = 2\theta$$

$$\pi(\theta|0) = \frac{P(0|\theta)\pi(\theta)}{P(0)} = 2(1-\theta)$$

Rozkład *a posteriori* $(n = 2)$

$$P(2) = \int_0^1 P(1|\theta)^2 \pi(\theta) d\theta = \int_0^1 \theta^2 d\theta = \frac{1}{3}$$

$$P(1) = 2\int_0^1 P(1|\theta) P(0|\theta) \pi(\theta) d\theta = 2\int_0^1 \theta(1-\theta) d\theta = \frac{1}{3}$$

$$P(0) = \int_0^1 P(0|\theta)^2 \pi(\theta) d\theta = \int_0^1 (1-\theta)^2 d\theta = \frac{1}{3}$$

$$\pi(\theta|2) = \frac{P(1|\theta)^2 \pi(\theta)}{P(2)} = 3\theta^2$$

$$\pi(\theta|1) = \frac{P(1|\theta) P(0|\theta) \pi(\theta)}{P(1)} = 3\theta(1-\theta)$$

$$\pi(\theta|0) = \frac{P(0|\theta)^2 \pi(\theta)}{P(0)} = 3(1-\theta)^2$$

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$$\int_0^1 x^{a-1}(1-x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\int_0^1 x^m (1-x)^l dx = \frac{m!l!}{(m+l+1)!}$$

Rozkład *a posteriori*

$$P(k) = \binom{n}{k} \int_0^1 P(1|\theta)^k P(0|\theta)^{n-k} \pi(\theta) d\theta$$
$$= \binom{n}{k} \int_0^1 \theta^k (1-\theta)^{n-k} d\theta = \frac{1}{n+1}$$

$$\pi(\theta|k) = \frac{\binom{n}{k} P(1|\theta)^k P(0|\theta)^{n-k} \pi(\theta)}{P(k)}$$
$$= (n+1)\binom{n}{k} \theta^k (1-\theta)^{n-k}$$

Estymator bayesowski

$$\hat{\theta}(k) = \int_0^1 \theta \pi(\theta|k) d\theta$$
$$= (n+1)\binom{n}{k} \int_0^1 \theta^{k+1} (1-\theta)^{n-k} d\theta$$
$$= (n+1)\binom{n}{k} \frac{(k+1)!(n-k)!}{(n+2)!}$$
$$= \frac{k+1}{n+2}$$

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Przykład (Model dwumianowy).
Niech $\theta_1 < \theta_2 \in (0,1)$. Model statystyczny

$$\{\{0,1,\ldots,n\},\{B(n,\theta),\theta \in \Theta = \{\theta_1,\theta_2\}\}\}$$

Zbiór decyzji

$$\mathcal{D} = \{d_1 = \{\theta = \theta_1\}, d_2 = \{\theta = \theta_2\}\}$$

Bayesowski obszar przyjęć decyzji d_2

$$\left\{x \in \mathcal{X} : \frac{p(x|\theta_2)}{p(x|\theta_1)} > \frac{\pi_1[L_{21} - L_{11}]}{\pi_2[L_{12} - L_{22}]}\right\}$$

$$\frac{\theta_2^k(1-\theta_2)^{n-k}}{\theta_1^k(1-\theta_1)^{n-k}} > \gamma$$

$$\left[\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right]^k > \gamma \left[\frac{1-\theta_1}{1-\theta_2}\right]^n$$

Reguła bayesowska

$$k > n\frac{\log\left[\frac{1-\theta_1}{1-\theta_2}\right]}{\log\left[\frac{\theta_2}{\theta_1}\right] + \log\left[\frac{1-\theta_1}{1-\theta_2}\right]} + \log\gamma$$

Niech $\pi_1 = \pi_2 = 0.5$. Funkcja straty $L(d, \theta)$

	$\theta = \theta_1$	$\theta = \theta_2$
d_1	0	1
d_2	1	0

Reguła bayesowska

$$k > n \frac{\log\left[\frac{1-\theta_1}{1-\theta_2}\right]}{\log\left[\frac{\theta_2}{\theta_1}\right] + \log\left[\frac{1-\theta_1}{1-\theta_2}\right]}$$

Ryzyko reguły: prawdopodobieństwo błędnej decyzji

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Niech $n = 100$, $\theta_1 = 0.05$, $\theta_2 = 0.15$
Decyzja d_2, jeżeli $k > 9.19$

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